Lecture 23:

Conditioning

Theorem 23.1. Suppose $\{N(s): s \geq 0\}$ is a Poisson Process with rate λ. If we condition on the event $\{N(t)=n\}$ for some $n \geq 1$; let $T_1, T_2, \cdots, T_n$ be the arrival times before t; let $U_1, U_2, \cdots, U_n$ be independent and uniformly distributed on $[0,t]$; arrange U_i into increasing order $U_{i_1} \leq U_{i_2} \leq \cdots \leq U_{i_n}$; then

(a). the vector $(T_1, T_2, \cdots, T_n)$ has the same distribution as $(U_{i_1}, U_{i_2}, \cdots, U_{i_n})$;

(b). the set of arrival times $\{T_1, T_2, \cdots, T_n\}$ has the same distribution as the set $\{U_1, U_2, \cdots U_n\}$;

this means without ordering

(c). if $0 \leq r < t$ and $0 \leq m \leq n$, then

$$\mathbb{P}(N(r)=m \mid N(t)=n) = \binom{n}{m} \cdot \left(\frac{r}{t}\right)^m \cdot \left(1-\frac{r}{t}\right)^{n-m}.$$

That is, the conditional distribution of $N(r)$ given $\{N(t)=n\}$ is $\text{binomial}(n, \frac{r}{t})$ and does not depend on λ.

Example 23.1. For instance, if $N(3)=4$, we have

$$\mathbb{P}(N(1)=1 \mid N(3)=4) = \binom{4}{1}\cdot\left(\tfrac{1}{3}\right)^1\left(1-\tfrac{1}{3}\right)^3 = \frac{32}{81}.$$

Example 23.2. Trucks and cars on highway 401 are Poisson Processes with rates 40 per hour and 100 per hour, respectively. $\frac{1}{8}$ of trucks and $\frac{1}{10}$ of cars get off on exit 365 to Allen Road.

Q(a): Find the probability that exactly 6 trucks arrive at Allen Road from Highway 401 between noon and 1pm.

A: By thinning, trucks arriving at Allen Road from Highway 401 follow Poisson Process with rate 5/hr. Thus, trucks arrive between noon and 1 pm has Poisson(5).

Therefore, the probability of interest is

$$e^{-5} \cdot \frac{5^6}{6!} .$$

Q(b): Given that there are 6 trucks arriving between noon and 1pm, what is the probability that exactly two arrived between 12:20 and 12:40?

A: Conditioning on $N(t)=6$, the probability of interest is

$$\binom{6}{2} \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4 .$$

Q(c): If we start watching at noon, what is the probability that four cars arrive before two trucks do?

A: That is, at least four cars arrive in the first 5 arrivals. Since the arrivals of cars follow Poisson Process with rate 10/hr and the arrivals of trucks follow Poisson Process with rate 5/hr, the arrivals of both follow Poisson Process with rate 15/hr with probability $\frac{10}{10+5} = \frac{2}{3}$ for each arrival being a car arrival. Thus, the probability of interest is

$$\left(\frac{2}{3}\right)^5 + \binom{5}{1}\cdot\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) = \frac{112}{243}.$$

Q(d): Suppose all trucks have 2 passengers while 30% of the cars have 1 passenger, 50% have 2, and 20% have 4. Find the mean and standard deviation of the number of passengers that arrive in two hours.

A: By thinning, the arrivals of trucks follow Poisson Process with rate 5/hr; the arrivals of cars with 1 passenger, 2 passengers, and 4 passengers follow Poisson Processes with rates 3/hr, 5/hr, and 2/hr, respectively. Therefore, these arrivals in two hours has distributions Poisson(10), Poisson(6), Poisson(10), and Poisson(4), respectively. Thus, by Theorem 21.1, the means and variances are $10 \cdot 2$, $6 \cdot 1$, $10 \cdot 2$, $4 \cdot 4$ and $10 \cdot 2^2$, $6 \cdot 1^2$, $10 \cdot 2^2$, $4 \cdot 4^2$, respectively. Since these arrivals are independent, the mean and variance of the total passengers are

$$10 \cdot 2 + 6 \cdot 1 + 10 \cdot 2 + 4 \cdot 4 = 62$$

and

$$10 \cdot 2^2 + 6 \cdot 1^2 + 10 \cdot 2^2 + 4 \cdot 4^2 = 150,$$

respectively. Thus, the deviation of interest is $\sqrt{150}$.

Method 2: Superposition.

By thinning, the arrivals of trucks follow Poisson Process with rate 5/hr; the arrivals of cars with 1 passenger, 2 passengers, and 4 passengers follow Poisson Processes with rates 3/hr, 5/hr, and 2/hr, respectively. By Superposition, the arrivals follow Poisson Process with rate 15/hr with probability $p_1 = \frac{5}{15} = \frac{1}{3}$, $p_2 = \frac{3}{15} = \frac{1}{5}$, $p_3 = \frac{5}{15} = \frac{1}{3}$, $p_4 = \frac{2}{15}$ of being in each category. Let Y_i be the number of passengers of the i-th arrival, then Y_i are i.i.d. and $\mathbb{P}(Y_i = 1) = p_2 = \frac{1}{5}$, $\mathbb{P}(Y_i = 2) = p_1 + p_3 = \frac{2}{3}$, and $\mathbb{P}(Y_i = 4) = p_4 = \frac{2}{15}$. Thus,

$$\mathbb{E}Y = 1 \cdot \frac{1}{5} + 2 \cdot \frac{2}{3} + 4 \cdot \frac{2}{15} = \frac{31}{15},$$

and

$$\mathbb{E}Y^2 = 1^2 \cdot \frac{1}{5} + 2^2 \cdot \frac{2}{3} + 4^2 \cdot \frac{2}{15} = 5.$$

Notice that the total number of arrivals in two hours N follows $\text{Poisson}(\lambda t) = \text{Poisson}(30)$, by Theorem 21.1,

$$\mathbb{E}S = 30 \cdot \mathbb{E}Y = 62,$$

and

$$\text{var}(S) = 30 \cdot \mathbb{E}Y^2 = 150.$$

This is the end of this lecture!